papers

- A. Bjorner, L. Lovasz, P. Shor,
  Chip-firing games on graphs, 1991
- A. Bjorner, L. Lovasz, chip-firing games on directed graphs, 1992
- P. Bak, How nature works,
  "The science of self organized criticality"

explain
- traffic jam
- forest fires
- $\boxed{\text{criticality}}$

# Monoids

(toward algebraic models associated to sandpile ...)

## References

- J. Rosales, P. García Sánchez
  finitely generated Commutative monoids
- F. Wehrung, Refinement monoids

**Def** $M$ non-empty set, $+ : M \times M \to M$
$(a,b) \mapsto a+b$

s.t. $a+b = b+a \quad \forall a,b \in M$

$(a+b)+c = a+(b+c) \quad \forall a,b,c$

$\exists \; 0 \in M$ s.t. $a+0 = 0+a = a$

$0$ is unique because $0 = 0+0' = 0'$

Ex $\mathbb{N}$, $\mathbb{N}\times\mathbb{N}$ monoids

| | 0 | a |
|---|---|---|
| 0 | 0 | a |
| a | a | 0 |

← two choice

$\mathbb{Z}_2$ group

| | 0 | a |
|---|---|---|
| 0 | 0 | a |
| a | a | a |

← $a+a=a$

Def (ordering) We say $a \leq b$ if $b = a + x$ for some $x \in M_0$ *

$\boxed{a<b \text{ if } a \leq b \text{ and } a \neq b}$

Our definition: $a<b$ if $b = a+x$ for some $x \neq 0$

With this definition in above example

$a<a$.

Question When (*) and (**) coincide?

$X$ set $\quad \mathcal{P}(X)$ power set

$(\mathcal{P}(X), \cup)$ and $(\mathcal{P}(X), \cap)$ Comm monoid

$A + B = A \cup B \qquad A \cdot B = A \cap B$

**Def** $M$ Commutative monoid

→ $M$ is conical if $a+b=0 \to a=b=0$

$C_2 = \{0, a\}$ conical

→ $M$ is cancellative if $a+b = a+c \to b=c$

$C_2 = \{0, a\}$ is not cancellative

– $M$ is refinement if $a, b, c, d \in M$

with $a+b = c+d$ then $\exists\ e_1, e_2, e_3, e_4 \in M$ s.t

$$\begin{array}{c|c|c|} & c & d \\ \hline a & e_1 & e_2 \\ \hline b & e_3 & e_4 \\ \hline \end{array}$$

$a = e_1 + e_2 \qquad c = e_1 + e_4$

$b = e_3 + e_4 \qquad d = e_2 + e_1$

free monoids are refinement

**Ex** $(\mathcal{P}(X), \cup)$ is refinement $\quad A + B = A \cup B$

## Group Completion

$M$ commutative monoid $\qquad (\mathbb{N} \rightsquigarrow \mathbb{Z})$

The group completion of $M$ is $\qquad \leftarrow$ free ~~monoid~~ abelian grp

$$G(M) = M^+ = \frac{F\langle [m] \mid m \in M \rangle}{\langle [m+n] - [m] - [n] \rangle}$$

taking into account the structure of $M$

## The property of group completion

$M$ com monoid, $G$ an abelian grp

$$\begin{tikzcd} M \arrow[r, "\theta"] \arrow[dr] & G \\ & M^+ \arrow[u, dashed, "\exists!"'] \end{tikzcd} \qquad m \mapsto [m]$$

proof $\quad \psi : M^+ \to G$

$\qquad [m] \mapsto \theta(m)$

well-defined and unique

## Another construction

$M$ comm. monoid

$$M^+ = M \times M / \sim \quad \text{where}$$

5

$(m_1, n_1) \sim (m_2, n_2)$ if $m_1 + n_2 + x = n_1 + m_2 + x$

observe $\sim$ is an equivalence relation such that

$M \times M/\sim$ is a group

$(m, m) \sim (n, n)$ and $[n, m]$ is zero $0$.

$[m, n] + (n, m) = 0.$

$$\boxed{M \times M/\sim \;\cong\; M^+}$$ Exercise

If $M$ is finite $\rightarrow$ $M \times M/\sim$ finite

but can you see $\dfrac{F\langle [m] \mid m \in M \rangle}{\langle [m+n] - [m] - [n] \rangle}$

that it is finite?

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M a comm monoid

$\sim$ eq relation which preserves the structure of monoid

$M/_\sim$ is a monoid $\quad [m] = \{x \in M \mid m \sim x\}$

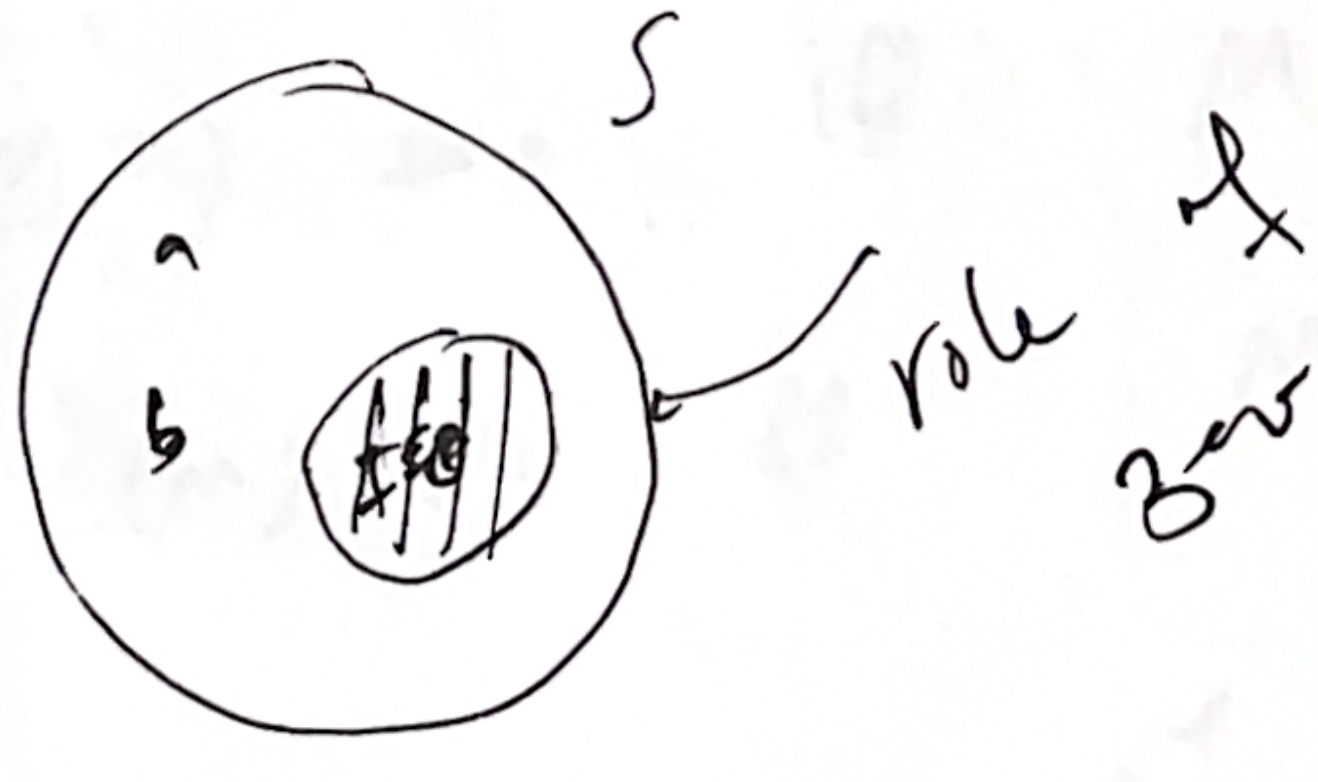
$M \to M/_\sim$

<u>Ex</u> <u>Rees quotient</u> $\quad S$ monoid

$I \subseteq S$ ideal meaning $a + I \subseteq I \ \forall a$

define congruence $a \sim b$ if $a = b$ or $a, b \in I$

Then $S/_I = \{ [x^\circ], a \mid \begin{array}{l} x \in I \\ a \in S/I \end{array} \}$




<u>Further quotients</u> $\quad M$ monoid
$N$ sub monoid $\quad \left( \begin{array}{l} 0 \in N \\ a, b \in N \to a+b \in N \end{array} \right)$

define $a \underset{N}{\sim} b$ if $a + n_1 = b + n_2$ for $n_1, n_2 \in N$

$\sim$ is a congruence of monoid

$M/N := M/\sim$ (The Grigorov equivalent to $a \sim b$ if $(a+N) \cap (b+N) \neq \phi$) ⑦

observe $n \sim 0 \quad \forall \; n \in N$

---

$M$ ~~monoid~~ commutative monoid

$Z(M) := \{ a \in M \mid a+b = 0 \text{ for some } b \in M \}$

called the unit group of $M$.

- $Z(M)$ is a group
- $Z(M) = 0$ iff $M$ is conical
- $Z(M) = M$ iff $M$ is an abelian group.

<u>Lemma</u> $\varphi : M \longrightarrow M^{+}$ is injective iff

$m \longmapsto [(m, 0)]$

$M$ is cancellative $\qquad M \times M/N$

Lemma  $M$ a finite commutative monoid.

The group completion $M^+$ is isomorphic to the smallest ideal of $M$.

($I \subseteq M$ is an ideal if $a + I \subseteq I$)

Proof  Set $I = \bigcap_{a \in M} (a + M)$

Since $M$ is finite, $t = \sum_{a \in M} a$ and $t \in I$.

so $I \neq \phi$. and $I$ is an ideal.

If $J$ an ideal then $I \subseteq J$ so $I$ is smallest

Exercise  prove $I$ is a group

in particular $\exists\, z \in I$ s.t. $i + z = i$ $\forall\, i \in I$

tricky

Defin $\varphi : M \longrightarrow I$ is a hom.

$m \longmapsto m + I$

check if has the universal property:

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\psi"'] & I \arrow[dl, dashed, "\psi|_I"] \\ G & \end{tikzcd}$$

is unique

Ex $M = \{0, u, 2u, 3u\}$ with $\boxed{4u = 2u}$

find $M^+$.

$u + M = \{u, 2u, 3u\}$

$2u + M = \{2u, 3u\}$

$3u + M = \{3u, 2u\}$

intersection $M^+ = \{2u, 3u\} \cong \mathbb{Z}/_2$

with zero $2u$

# Graph monoids

**Recall** a directed graph $E = (E^0, E^1, r, s)$

$e \in E^1$ is a *loop* if $r(e) = s(e)$

$v \in E^0$ is a sink if $|s^{-1}(v)| = 0$

$v \in E^0$ is *irrelevant* if $|s^{-1}(v)| = 1$

$p$ a path $\quad p = e_1 e_2 e_3 \cdots e_n$ s.t. $r(e_i) = s(e_{i+1})$

$\xrightarrow{e_1}\xrightarrow{e_2}\xrightarrow{e_3}\cdots\xrightarrow{e_n}$

[ if you think categorically then $e_n e_{n-1} \cdots e_2 e_1$

**Def** (weighted graph) A weighted graph $(E, w)$ is a graph $E$, $w : E^1 \to \mathbb{N}^+$,

so $e \in E^1$, $w(e)$ the weight of $e$

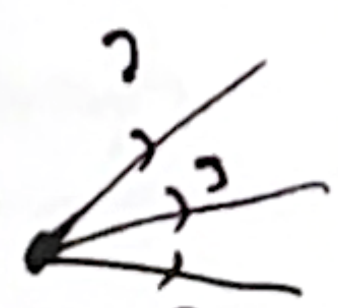

$v \in E^0$, $w(v) = \max\{ w(e) \mid e \in s^{-1}(v) \}$

(11)

A vertex weighted graph if ~~each weight~~ for each $v \in E^\circ$ all weights of edges are equal one for $v \in E$; and

for $e_1, e_2 \in s^{-1}(v) \to w(e_1) = w(e_2)$.

A balanced weighted graph if it is vertex weighted and $v(x) = |s^{-1}(x)|$ or $\forall e \in s^{-1}(v)$, $w(e) = |s^{-1}(e)|$



Ex

balanced weighted graph.

Def (Hereditary, saturated subset)

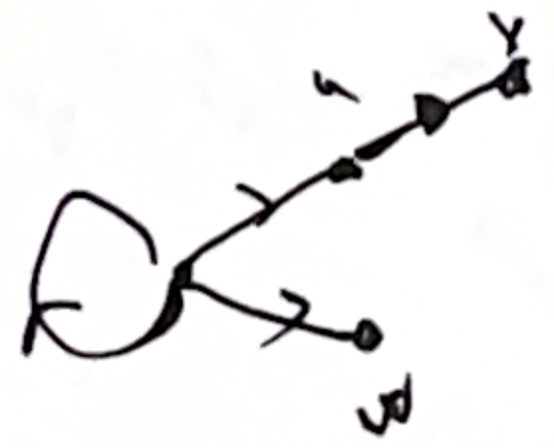
Her $H \subseteq E^\circ$ her if for any $e \in E'$, if $s(e) \in H$

Then $r(e) \in H$.

Sat $H$ is sat if for $v \in E^\circ$ s.t. $r(s^{-1}(v)) \subseteq H$

Then $v \in H$

Example $E$ a graph $S = \{$ all vertices which do not connect to any cycle $\}$

$E$: 

$H = \{u, v, w\}$ her & sat

<u>Quotient graph</u> $H \subseteq E^0$, then

$$(E/H)^0 = E^0 \setminus H^0$$

$$(E/H)^1 = \{e \in E \mid r(e) \notin H\}$$

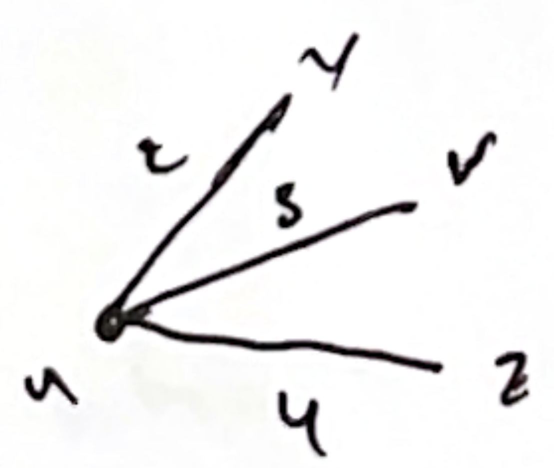
in the above example $E/H = $ ○

<u>Graph monoid</u>

<u>Def</u> $(E, w)$ weighted graph, assign $w(v) = 1$ for sinks

The <u>reduced graph monoid</u> of — free monoid on $E^0$

$$M(E,w) = \frac{\mathbb{F}_E}{\left\langle w(v)v = \sum_{e \in s^{-1}(v)} r(e) \mid v \in E^0 \right\rangle}$$



$w(u) = 4$ $\boxed{4u = x + w + z}$

$v$ sink $w(v) = 1$ $v = \sum_{\varphi} = 0$

<u>Exmp</u> 

$w(e_i) = m$

$$M(E,w) = \frac{\mathbb{F}v}{\langle mv = nv \rangle} \cong \frac{\mathbb{N}}{\langle m = n \rangle}$$

$C_{n,m} = \{0, 1, 2, \ldots, m, m+1, \ldots, m+n\}$ $n$



M/N

Congruence defined on $M(E, w)$

Consider $\mathbb{F}_E$. define $\sim$ $r \leftarrow$ transformation

set $r(w(v)v) = \sum_{e \in s^{-1}(v)} r(e)$

non-zero element of $\mathbb{F}_E \to$ uniquely up to permutation $\sum_{i=1}^{n} k_i v_i$

$v_i$ distinct, $k_i \in \mathbb{N}^+$

Define binary relation $\to_1$ on $\mathbb{F}_E$

$$\sum k_i v_i \mapsto_1 \left(\sum_{i \neq j} k_i v_i\right) + \left(k_j - w(v_j)\right) v_j + r\left(w(v_j) v_j\right)$$

when $1 \leq j \leq n$ and $k_j \geq w(v_j)$

( I am just doing sandpile!!)

Let $\to$ be the transitive and reflexive closure (15)

of $\to_1$ on $\mathbb{F}_E$ i.e

$a \to b$ if $a = b$ or $a = a_0 \to_1 a_1 \to_1 \cdots \to_1 a_k = b$

Finally $\sim$ be the congruence on $\mathbb{F}_E$ generated by

$\to$ i.e

$a \sim b$ if $a = a_0, a_1, \ldots, a_n = b$ in $\mathbb{F}_E$

such that $a_i \to a_{i+1}$ or $a_{i+1} \to a_i$

$a \to \to \leftarrow \leftarrow \to \leftarrow \to \to b$

then $M(E, w) = \mathbb{F}_E / \sim$.

Fact $E$ sandpile graph. Consider $E$ as a balanced weighted graph (so $w(v) = |s^{-1}(v)|$ for non-sink) ($w(v) = 1$ for sink) Then $SP(E) = M(E, w)$.

Confluence property

The monoid $M(E, w)$ has a "Confluence property" i.e $a, b \in \mathbb{F}(E, w)$ and $a = b \in M(E,w)$ then $\exists\, c$

such that

$$\begin{tikzcd} a \arrow[dr, dashed] & & b \arrow[dl, dashed] \\ & c & \end{tikzcd}$$

$a \rightarrow \leftarrow \rightarrow b$

proof

Corollary $(E, w)$ weighted graph.

$H \subseteq E$ hereditary


$\varphi : M(H, w) \longrightarrow M(E, w)$
$a \longmapsto a$

well defined map. $\varphi$ is injective.

proof $a \longmapsto \varphi(a) = a$ Now in $M(E, w)$
$b \longmapsto \varphi(b) = b$

$$\begin{tikzcd} a \arrow[dr, dashed] & & b \arrow[dl, dashed] \\ & c & \end{tikzcd}$$

Since $H$ is hereditary all

vertices in $v(\cdot w(v) v) = \sum_{e \in s^{-1}(v)} r(e)$ appears in $H$

so $a \sim b$ in $H$. i.e $a = b$ in $M(H, w)$.

Prop $(E,w)$ finite weighted graph

$M(E,w)$ reduced monoid

$S$ set of vertices in connected to any

cycle. Then $Z(M(E,w)) \cong M(S,w)$

Furthermore

- $M(E,w)$ is a group iff $E$ is acyclic
- $M(E,w)$ is conical iff every vertex in $S$ has weight one

Finally $(E,w)$ finite vertex graph

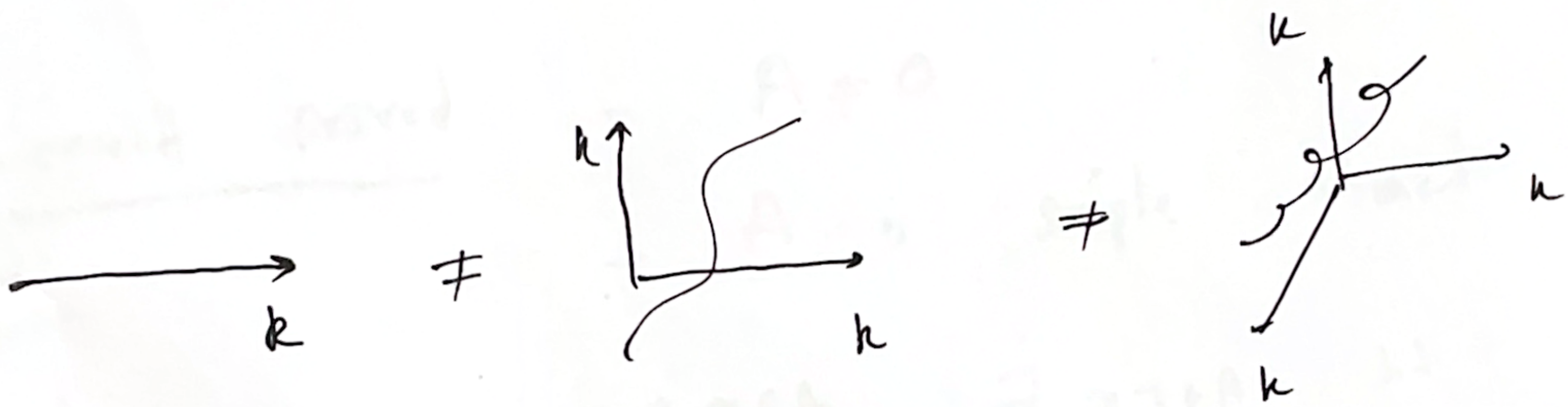
$$M(E,w) / M(H,w) \cong M(E/H, w)$$

# Leavitt path algebras

**History** A vector space has a unique dimension

$$V/k \text{ field} \Rightarrow V \cong k^n \qquad \dim V = n$$

$\hookrightarrow$ i.e $k^n \cong k^m$ as $k$-vector space $\to n = m$




instead of $k$ replace by a unital ring

$A^n \cong A^m$ as $A$-module $\longrightarrow$ $\begin{cases} A \text{ Comm} \to n = m \\ A \text{ "finite cond"} \Rightarrow n = m \end{cases}$

Comm ring $A \longrightarrow A/m$ field, if $A^n \cong A^m$

$A^n \otimes_A A/m \cong A^m \otimes_A A/m \to (A/m)^n \cong (A/m)^m$

$\Rightarrow n = m$

## Leavitt's Construction

free unital algebra

$$A = \frac{K\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle}{\left\langle (x_1, \dots, x_n)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 1, \ \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}(x_1, \dots, x_n) = I_{n\times n} \right\rangle}$$

Undergraduate exercise

$$A \cong A^n \text{ as } A\text{-module}$$

$$A \xrightarrow{\varphi} A^n \qquad , \qquad A^n \xrightarrow{\psi} A$$

$$a \longmapsto (ax_1, \dots, ax_n) \qquad , \qquad (a_1, \dots, a_n) \longmapsto a_1y_1 + \dots + a_ny_n$$

$$\rightarrow \varphi\psi = id \; , \; \psi\varphi = id$$

Leavitt proved

- $A \neq 0$
- $A$ is simple infact

$0 \neq a \in A \; , \; \exists \; x, y \in A$ s.t. $xay = 1$

- $A \ncong A^2 \ncong A^3 \ncong \dots \ncong A^n$

$A^i \ncong A^j \quad 1 \leq i, j \leq n-1$

Set $n = 2$

$$A = \frac{\langle x_1, x_2, y_1, y_2 \rangle}{\left\langle \begin{array}{c} x_1y_1 + x_2y_2 = 1 \\ y_1x_1 = y_2x_2 = 1 \end{array} \right\rangle}$$

Construct

**Project** Describe all simple modules of $L_2$

( Chen, Li, Rangaswamy, Ara, Abrams,

Steinberg, ...... )

**Back to Leavitt**

$$A = \frac{K\langle x_{ij},\ y_{ji},\ \ 1\le i\le n,\ 1\le j\le m\rangle}{\left\langle \begin{pmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & & \\ x_{n1} & \cdots & x_{nm} \end{pmatrix} \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{m1} & & y_{mn} \end{pmatrix} = I \right\rangle}$$

$$\begin{pmatrix} Y \end{pmatrix} \begin{pmatrix} X \end{pmatrix} = 1$$

**Leavitt proved**

- $A \neq 0$
- $A$ is a domain !!
- $A \not\cong A^2 \not\cong \cdots \not\cong \underbrace{A^m \not\cong \cdots \not\cong A^{n-1} \cong A^n}_{\cong}$

④

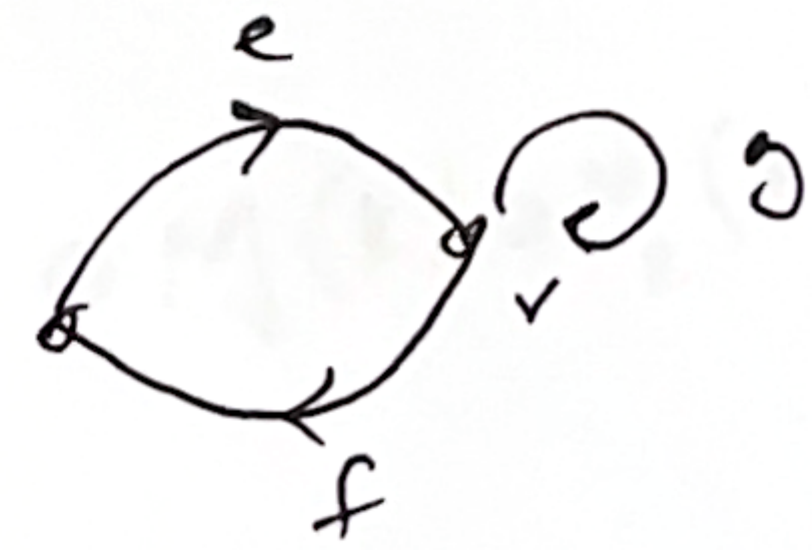
## Leavitt path algebras

– Abrams – Aranda Pino

– Ara – Moreno – Pardo     2005

$E$ directed graph

$E = (E^0, E^1, r, s)$




$E^0 = \{u, v\}, \quad E^1 = \{e, f, g\}$

**Def** $E$ directed graph, $k$ a field,

$$L_k(E) := \frac{k\langle v, e, e^* \mid v \in E^0, e \in E^1 \rangle}{\ }$$

← free non-commutative algebra

path algebra

$uv = \delta_{u,v} v \quad$ for $u, v \in E^0$

$s(e) e = e\, r(e) = e \quad$ for $e \in E^1$

$r(e) e^* = e^* s(e) = e^*$

$\sum_{e \in s^{-1}(v)} ee^* = v \quad$ for $v$ regular

$e^* e' = \delta_{e,e'} r(e) \quad e, e' \in E^1$

## Exercise

1. $n$-loops

$L_k($  $) \cong L_n$

non-unital free algebra

free unital algebras

2. $L_K($  $) \cong M_2(K) \oplus M_2(K) \oplus M_2(K)$

3. $L_k($  $) \cong M_2(K[x, x^{-1}])$

4. $L_K($  $) = !?$

Since 2005 $\rightsquigarrow$ three hundereal papers...

graph $E$ has property $Q$ $\longleftrightarrow$ $L_K(E)$ has property $P$

First theorem published

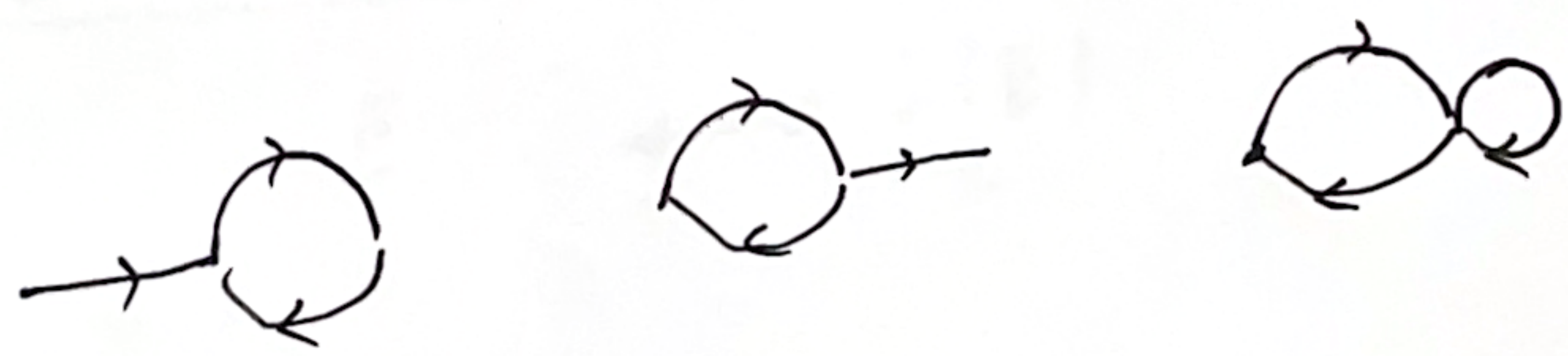


th (Abrams, Aranda Pino)

E a finite graph, k field, $L_k(E)$ the Leavitt path algebra then

$L_k(E)$ is a simple ring iff

1. Every vertex connects to every cycle and every sink
2. Every cycle has an exit

Ex 

Exer If $L_k(E)$ is simple and E has sink

$\iff$ E has one sink, and is acyclic

Def A ring R is called "Zorn" if every element $a \in R$ is either nilpotent ($a^n = 0$, $\exists n$) or $\exists\, b \in R$ s.t. $ab \neq 0$ and idempotent (could be both)

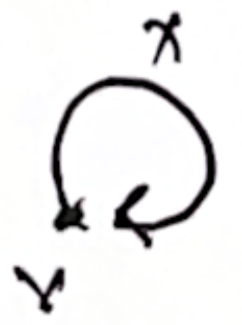
(Rangaswamy)



**Th** $E$ finite graph, $k$ a field.

Then $L_k(E)$ is Zorn iff every cycle has an exit

**Proof** $\boxed{\text{Exercise}}$ Suppose $L_k(E)$ is Zorn $\Longrightarrow$ 



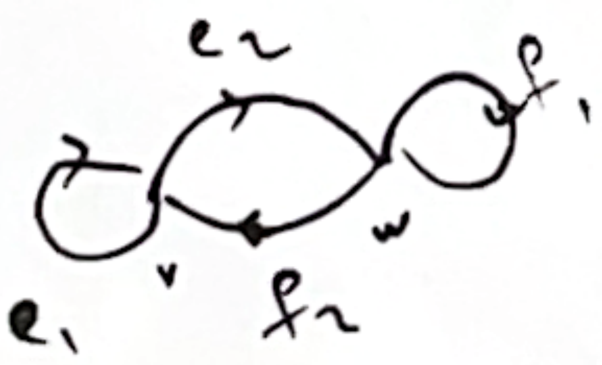

$K[x, x^{-1}]$ $\quad 1+x$ → not nilpotent ; → $(1+x)a$ idempotent

**Many many more interesting theorems**

$$E \longleftrightarrow L_k(E)$$

$E$ 


$\xrightarrow{\text{one can prove}}$ $L_k(E) \cong L_k(F)$

$F$ 


**Exercise**

$$L_k(E) \longrightarrow L_k(F)$$
$$u \longmapsto v+w$$
$$e \longmapsto e_1+e_2$$
$$f \longmapsto f_1+f_2$$

Show isomorphism

**So big question** $E, F$ graphs

when $L_k(E) \cong L_k(F)$ $\quad$ classification